Surface Roughness and Effective Stick-Slip Motion

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The effect of random surface roughness on hydrodynamics of viscous incompressible liquid is discussed. Roughness-driven contributions to hydrodynamic flows, energy dissipation, and friction force are calculated in a wide range of parameters. When the hydrodynamic decay length (the viscous wave penetration depth) is larger than the size of random surface inhomogeneities, it is possible to replace a random rough surface by effective stick-slip boundary conditions on a flat surface with two constants: the stick-slip length and the renormalization of viscosity near the boundary. The stick-slip length and the renormalization coefficient are expressed explicitly via the correlation function of random surface inhomogeneities. The effective stick-slip length is always negative signifying the effective slow-down of the hydrodynamic flows by the rough surface (stick rather than slip motion). A simple hydrodynamic model is presented as an illustration of these general hydrodynamic results. The effective boundary parameters are analyzed numerically for Gaussian, power-law and exponentially decaying correlators with various indices. The maximum on the frequency dependence of the dissipation allows one to extract the correlation radius (characteristic size) of the surface inhomogeneities directly from, for example, experiments with torsional quartz oscillators.

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I. INTRODUCTION

Progress in micro- and nanotechnology requires better understanding of boundary effects. For hydrodynamic microflows, this means better understanding of stick-slip motion near solid walls and, in particular, information on the dependence of the slip (or stick) length on the properties of the walls. Despite the fact that similar issues were first raised more than a hundred years ago [1, 2, 3], the slip length remains one of the least known transport coefficients. Traditionally, the most detailed information on the boundary slip is available for rarefied classical gases[4, 5, 6, 7] in applications to vacuum technology, high altitude flights, and space research. More recently [8, 9, 10, 11, 12, 13, 14, 15], liquid ^3He has become an important source of information on surface slip. This is not surprising since, in contrast to classical gases, one can easily vary the quasiparticle mean free path in ^3He by changing temperature thus allowing experiments in a wide range of Knudsen numbers.

The conventional theory of boundary slip assumes that the slip length is proportional to the bulk mean free path $\mathcal{L}_b$, $\mathcal{L}_{sl} = \alpha\mathcal{L}_b$, and ignores small-scale surface inhomogeneities. Obviously, this approximation is too crude. The hydrodynamic flows near the walls strongly depend on geometry of surface inhomogeneities [16]. Recent analysis of slip near a model surface with periodic irregularities demonstrated [17] that the effective slip length $\mathcal{L}_{eff}$ contains not only the bulk component $\alpha\mathcal{L}_b$ but also the contribution from the averaged surface curvature R, $\mathcal{L}_{eff}^{-1} = \alpha^{-1}\mathcal{L}_b^{-1} - R^{-1}$. An application of the corresponding boundary condition to several types of curved walls [18] resulted in an interesting expression for an effective slip length which could, under certain circumstances, be equivalent to large-scale surface roughness. However, the results [17, 18] were obtained for few special types of regular surface inhomogeneities only. In the case of micro- and nanoscale defects, it is more realistically to suggest that surfaces have *random* corrugation. What is more, in some cases, especially in the hydrodynamic limit $\mathcal{L}_b \to 0$, it is not clear how to use the effective boundary parameters of Refs. [17, 18].

Below we derive an effective stick-slip boundary condition which would reproduce hydrodynamic flows with $\mathcal{L}_b = 0$ near rough walls with small-scale *random* inhomogeneities. Since the hydrodynamic calculations near inhomogeneous walls are extremely complicated [16], it is highly desirable to map this problem onto the system with simple flat surface geometry with some effective boundary condition. This boundary condition should contain information about geometrical and statistical properties of the real corrugated surface and ensure a proper behavior of hydrodynamic

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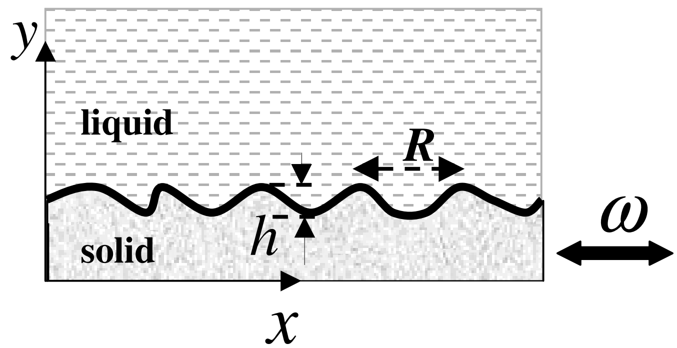

FIG. 1: General geometry of the model.

variables. The derivation of this simple boundary condition is the main goal of the paper. Below we show that this boundary condition contains two effective parameters: the effective stick-slip length and renormalized viscosity. We also demonstrate that the results for attenuation in torsional oscillator experiments can provide valuable information about the statistical type of surface inhomogeneities and give the values of the main geometrical parameters of surface roughness.

In the next Section, we present the main hydrodynamic equations and find the stream function in systems with random rough walls (details of the derivation are given in Appendix A). Comparison of the hydrodynamic results with those for the stick-slip motion allow us to get the expression for the effective stick-slip parameters in Section III. For clarification of the physical meaning of the parameters in the somewhat unexpected effective boundary condition, we present a simple hydrodynamic model for a boundary layer in Appendix B. Section IV contains analytical and numerical results for surfaces with various statistical types of inhomogeneities. Summary and conclusions are presented in Section V.

II. HYDRODYNAMIC FLOWS ALONG ROUGH WALLS

To determine an effective slip/stick length, one has to solve an appropriate hydrodynamic problem with a boundary condition on a random rough wall and to compare the results with those for a similar problem with a slip boundary condition on a smooth wall. Several "typical" hydrodynamic problems [19] have been generalized recently in order to cover boundaries with slight roughness [20, 21, 22, 23, 24]. For our purposes, the most appropriate problem is the problem of hydrodynamic flows excited by tangential oscillations of a rough wall. The advantages are the convenience of the experimental setup with a standard transverse oscillator, a choice of several observables such as hydrodynamic velocity and two components of the shear impedance, and the presence of an extra variable - frequency ω - that allows one to vary the ratio of the hydrodynamic decay length to the size of wall inhomogeneities. Since this problem has already been studied in Ref. [20], though by a different method, we will only briefly outline our hydrodynamic formalism in Appendix A and present some additional results.

We consider semi-infinite viscous fluid restricted by a rough solid wall. For simplicity, roughness is assumed one-dimensional with profile described by a random function $Y = \Xi\left(X\right)$ with the zero mean value, $\langle \Xi\left(X\right)\rangle = 0$. The wall is homogeneous in Z-direction (see Fig. 1). This inhomogeneous surface is characterized by two length parameters: the average amplitude h and correlation radius (size) R of surface inhomogeneities. We consider the case of slight roughness,

$$\epsilon = h/R \ll 1. \tag{1}$$

In other situations, any general description of hydrodynamic flows near rough walls is virtually impossible.

The wall oscillates in X-direction with the velocity

$$\mathbf{U}\left(t\right) = \mathbf{e}_x u_0 \cos\left(\omega t\right) \tag{2}$$

The hydrodynamic boundary condition is the condition of zero velocity $\mathbf{V}$ on the wall in the reference frame in which the wall is at rest:

$$\mathbf{V}\left(X - \int \mathbf{U}\left(t\right) dt, Y = \Xi\left(X - \int \mathbf{U}\left(t\right) dt\right)\right) = \mathbf{0}, \tag{3}$$

Two important hydrodynamic length scales are the decay length (or the viscous wave penetration depth), δ, and the amplitude of the boundary oscillations, a,

$$\delta = \sqrt{2\nu/\omega},\ a = u_0/\omega, \tag{4}$$

where $\nu = \eta/\rho$ is the kinematic viscosity.

It is convenient to choose h, R and the amplitude of the wall velocity u_0 as the scaling parameters and introduce dimensionless variables as

$$\mathbf{v} = \mathbf{V}/u_0,\ x = X/R,\ y = Y/R,\ \xi(x) = \Xi(X/R)/h. \tag{5}$$

When the fluctuations of $\xi(x)$ are statistically independent and the higher momenta can be expressed through the second one, the random surface roughness is actually described not by the unknown random function $\xi(x)$ with the zero average, but by the correlation function $\zeta(x)$:

$$\zeta(x) \equiv \langle \xi(x_1)\,\xi(x_1+x)\rangle = \frac{1}{A}\int_{-\infty}^{\infty} \xi(x_1)\,\xi(x_1+x)\,dx_1, \tag{6}$$
$$\langle \xi(k_x)\,\xi(k_x')\rangle = 2\pi\delta(k_x+k_x')\,\zeta(k_x),$$

where A is dimensionless flat surface area of the wall. Experimentally the correlation functions $\zeta(x)$ [or its Fourier image, also called the power spectrum, $\zeta(k_x)$] can exhibit different types of long-range behavior and assume various forms [25]. Particular examples of the surface correlators are analyzed in Section IV. Note, that in our dimensionless notations (5) the correlation radius of the surface inhomogeneities is equal to 1.

The liquid is considered incompressible, $\operatorname{div}\mathbf{v} = 0$. In variables (5), the dimensionless Navier-Stokes equation can be written as

$$\frac{1}{\omega_0}\frac{\partial \operatorname{rot}\mathbf{v}}{\partial t} - \nabla^2 \operatorname{rot}\mathbf{v} = \Re\left[(\operatorname{rot}\mathbf{v}\nabla)\,\mathbf{v} - (\mathbf{v}\nabla)\operatorname{rot}\mathbf{v}\right] \tag{7}$$

where the characteristic frequency ω_0 and the Reynolds number $\Re$ are

$$\omega_0 = \frac{\nu}{R^2},\ \Re = \frac{u_0 R}{\nu} \equiv \frac{a}{R}\frac{\omega}{\omega_0} \tag{8}$$

(the inverse frequency parameter ω_0^{-1} is often called the diffusion time of vorticity) Since the first term in Eq. (7) has an order of $(\omega/\omega_0)\operatorname{rot}\mathbf{v}$, the hydrodynamic flows are characterized by the dimensionless parameter

$$\Lambda = \sqrt{\omega/\omega_0} = \sqrt{2}R/\delta \tag{9}$$

which describes the ratio of the size of inhomogeneities R to the hydrodynamic decay length δ. Two dimensionless parameters, ϵ and Λ, are the main parameters of the problem.

Below we consider the linearized Navier-Stokes equation without the nonlinear term in *r.h.s.* of Eq. (7). For small frequencies, $\omega/\omega_0 \ll 1$, this linearization is justified for small Reynolds numbers $\Re \ll 1$. In the opposite limit of high frequencies, $\omega/\omega_0 \gg 1$, this requires smallness of the amplitude of oscillations a in comparison with the tangential size of surface inhomogeneities R at arbitrary Reynolds numbers $\Re$, $a/R \ll 1$ [19]. The linearized Eq. (7) for $\operatorname{rot}\mathbf{v}$ can be, as usual, rewritten as the fourth order differential equation for the scalar stream function $\psi(x,y)$,

$$v_x = \frac{\partial\psi}{\partial y},\ v_y = -\frac{\partial\psi}{\partial x}. \tag{10}$$

In our problem, all hydrodynamic variables contain harmonic time dependence. After the transformation to the coordinate frame oscillating with the wall, the hydrodynamic equations and boundary conditions for the stream function acquire the form

$$-i\Lambda^2\nabla^2\psi - \nabla^4\psi = 0, \tag{11}$$
$$\frac{\partial\psi(x,\epsilon\xi(x))}{\partial y} = 1,\ \frac{\partial\psi(x,\epsilon\xi(x))}{\partial x} = 0, \tag{12}$$
$$\psi(x,\infty) = const. \tag{13}$$

The solution of the linearized Navier-Stokes equation (11) – (13) is quite difficult because the boundary condition (12) involves the rough wall with random inhomogeneities. Using a coordinate transformation $y \to y - \xi(x)$, we can reduce the Navier-Stokes equation to an equivalent equation with the boundary condition on the perfect flat wall. However, this new equation, as a result of the transformation-driven change in derivatives, acquires several additional terms $\widehat{V}\psi$ that involve the combinations of derivatives of ψ and the random function $\xi(x)$. To deal with these terms, we find the explicit form of the Green's function with the proper boundary condition. Then the problem reduces to a rather transparent integral equation

$$\psi(k_x, y) = \psi_{inh}(k_x, y) + \int_0^\infty dy'\, G(k_x, y, y') \int_{-\infty}^{\infty} \frac{dk'_x}{2\pi} \widehat{V}(k_x - k'_x, y')\, \psi(k'_x, y'). \tag{14}$$

This procedure and the explicit expressions for the unperturbed inhomogeneous solution $\psi_{inh}(k_x, y)$, the perturbation $\widehat{V}$, and the Green's function are given in Appendix A. In some sense, we shifted the difficulty from the boundary condition to the bulk equations with random sources of the special form. Note, that Eq. (14) is still exact and, in principle, could be solved without the perturbation theory. The explicit form of Green's function is such that one can extract the main part of the solution in the closed form. Another possible approach to Eq. (14) is to apply the Wiener-Hermite functional expansion [26, 27].

Here we solve Eq. (14) by iterations as an expansion in the small parameter: $\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots$ The first three terms for the stream function have the following forms

$$\psi_0(k_x, y) = \frac{2\pi}{i\lambda}\delta(k_x)\exp(i\lambda y), \tag{15}$$

$$\psi_1(k_x, y) = \xi(k_x)\left[e^{i\lambda y} + \frac{i\lambda}{s_2 - s_1}\left(e^{s_1 y} - e^{s_2 y}\right)\right], \tag{16}$$

$$\langle\psi_2(k_x, y)\rangle = \delta(k_x)\int_{-\infty}^{\infty} dk'_x\, \zeta(k'_x)\left[\frac{s_1 + s_2}{i\lambda}\left(i\lambda e^{i\lambda y} + s_1 e^{s_1 y} - s_2 e^{s_2 y}\right)\right], \tag{17}$$

where we exclude uninteresting constant terms and

$$\begin{aligned} s_1 &= -|k_x|,\ s_2 = \sqrt{k_x^2 - i\Lambda^2} \equiv -\alpha + i\beta, \\ \alpha, \beta &= \frac{1}{\sqrt{2}}\sqrt{\sqrt{k_x^4 + \Lambda^4} \pm k_x^2} \geq 0, \\ \lambda &= e^{i\pi/4}\Lambda. \end{aligned} \tag{18}$$

Since for further calculations we need only the expression for ψ_2 which is averaged over the random surface inhomogeneities, Eq. (17) gives only the compact expression for $\langle\psi_2(k_x, y)\rangle$.

These expressions for the stream function provide the roughness-driven corrections for the velocity and rate of energy dissipation (see Appendix A):

$$\langle v_x\rangle = \mathrm{Re}\left\{e^{i(\lambda y - \omega t)}\left[1 + i\lambda\epsilon^2\ell_1\right]\right\},\ \langle v_y\rangle = 0, \tag{19}$$

$$\ell_1 = \int_0^\infty \frac{dk_x}{\pi}\zeta(k_x)\left\{s_1 + s_2 - i\lambda/2\right\} \tag{20}$$

$$Q = -\frac{\eta u_0^2}{2R}\frac{\Lambda}{\sqrt{2}}\left[1 + \epsilon^2\Lambda^2\ell_2\right], \tag{21}$$

$$\ell_2 = \int_0^\infty \frac{dt}{\pi}\zeta(t\Lambda)\,\phi(t),\ \phi(t) = 1 - \sqrt{\sqrt{1 + t^4} - t^2}. \tag{22}$$

The equation for the energy dissipation is averaged over both the surface roughness and the period of oscillations. This expression is similar to the result of Ref. [20] obtained with the help of the Rayleigh perturbation method.

Stream function also allows one to find corrections to the roughness-driven friction force. These calculations should be done more carefully than for standard flat geometry: the friction force is parallel to the actual surface and, in the

case of the oscillating *rough* wall, has both components F_x and F_y. One should also take into account the y-component of velocity, which is absent in the case of flat geometry. Straightforward calculation for the averaged square of absolute value of dimensionless friction force give

$$\mathbf{F} = \frac{\eta u_0}{R}\mathbf{f},\ \left\langle \overline{f^2} \right\rangle = \frac{\Lambda^2}{2}\left[1 + \frac{\epsilon^2\Lambda^2}{\pi}\int_0^\infty dt \zeta(\Lambda t)\phi^2(t)\right]. \tag{23}$$

This expression is different from a simple experimental definition of the effective friction force $F_{eff} = -Q/u_0$

At low frequencies (large decay lengths, $\Lambda \ll 1$), Eqs. (19), (21) expressions for parameters $\ell_{1,2}$ reduce to

$$\ell_1 = -2\int_0^\infty \frac{dk_x}{\pi}\zeta(k_x)k_x + O(\Lambda), \tag{24}$$

$$\ell_2 = 1 + O(\Lambda \ln \Lambda), \tag{25}$$

and the equations for the velocity and attenuation acquire the following form:

$$\langle v_x \rangle = \mathrm{Re}\left\{e^{i(\lambda y - \omega t)}\left[1 - \Lambda e^{i3\pi/4}\epsilon^2\left(2\int_0^\infty \frac{dk_x}{\pi}\zeta(k_x)k_x + O(\Lambda)\right)\right]\right\}, \tag{26}$$

$$Q = -\frac{\eta u_0^2}{2R}\frac{\Lambda}{\sqrt{2}}\left[1 + \Lambda^2\epsilon^2(1 + O(\Lambda \ln \Lambda))\right]. \tag{27}$$

The fact that the main term in ℓ_2 is equal to 1 is due to our choice of the normalization of the correlation function in Eq. (5) as $\zeta(x=0) = 1$ (see also Section IV).

In the opposite limit of high frequencies $\Lambda \gg 1$,

$$Q \approx -\frac{\eta u_0^2}{2R}\frac{\Lambda}{\sqrt{2}}\left[1 + \frac{\epsilon^2}{2}\int_0^\infty \frac{dk_x}{\pi}\zeta(k_x)k_x^2\right] \equiv Q_0\left[1 + \frac{\epsilon^2}{2}\left\langle \xi'^2 \right\rangle\right]. \tag{28}$$

This result has a simple physical explanation. In this limit, the decay length is much smaller than correlation radius (size) of the wall inhomogeneities R. As a result, the dissipation occurs in a very narrow layer near the wall within which the wall can be considered as almost flat. Then the correction to dissipation stems simply from the increase in the surface area relatively to the flat boundary

$$Q \approx -\frac{\eta u_0^2}{2R}\frac{\Lambda}{\sqrt{2}}\frac{1}{L^2}\oint dA = -\frac{\eta u_0^2}{2R}\frac{\Lambda}{\sqrt{2}}\frac{1}{L}\int \sqrt{1+\epsilon^2\xi'^2}dx \tag{29}$$

Equation (28) is simply the combination of the first two terms in the Taylor expansion of Eq. (29) in small ϵ.

In principle, it is possible to slightly modify our problem by considering a torsional quartz crystal oscillator with density ρ_s, thickness d. If such a resonator has a rough solid-fluid interface, the frequency shift $\delta\omega$ of the resonance frequency Ω_0 acquires an additional roughness-driven component which can be described within the above formalism and should be given by the similar equations. Such a frequency shift for a transverse oscillator is [20]

$$\delta\omega = -\frac{\eta}{\sqrt{2}}\frac{\Lambda}{R}\frac{1}{\rho_s d}\left\{1 + \epsilon^2\Lambda\int_0^\infty \frac{dk_x}{\pi}\zeta(k_x)\left[\sqrt{\sqrt{k_x^4+\Lambda^4}+k_x^2} - \Lambda + \sqrt{2}k_x\right]\right\}.$$

We do not want to dwell on this issue; our interest in focused mainly on the roughness-driven corrections to the hydrodynamic flows and dissipation.

III. EFFECTIVE STICK-SLIP BOUNDARY CONDITIONS

The main aim of the paper is to find when and to what extent flows near random rough surface are equivalent to stick-slip motion with some effective stick-slip boundary conditions near flat surfaces,

$$\mathrm{Re}\left\{v_x(x,0,t) - \frac{\mathcal{L}_{eff}}{R}\frac{\partial v_x(x,0,t)}{\partial y}\right\} = \mathrm{Re}\left(e^{-i\omega t}\right) \tag{30}$$

where the effective stick-slip length $\mathcal{L}_{eff}$, in order to simplify the applications of the results, is introduced with the proper dimensionality of length while all other variables are still dimensionless, Eq. (5). With this boundary condition on a flat wall, the velocity field is

$$v_x\left(y,t\right) = \mathrm{Re}\left[\frac{e^{i(\lambda y-\omega t)}}{1-e^{i3\pi/4}\Lambda\mathcal{L}_{eff}/R}\right] \tag{31}$$

Since the roughness-generated corrections for velocity are small, the comparison between Eq.(31) and Eqs.(19) , (26) is possible only when $\Lambda\mathcal{L}_{eff}/R \ll 1$, *i.e.*, only for relatively large decay lengths (low frequencies),

$$v_x\left(y,t\right) \approx \mathrm{Re}\left[e^{i(\lambda y-\omega t)}\left(1+e^{i3\pi/4}\Lambda\mathcal{L}_{eff}/R\right)\right] \tag{32}$$

In this case, the comparison with the roughness-driven correction for the velocity at low frequencies, Eq. (26), yields the following simple expression for the effective stick-slip length $\mathcal{L}_{eff} = R\epsilon^2\ell_1$:

$$\mathcal{L}_{eff} = -2\frac{h^2}{R}\int_0^\infty \frac{dk_x}{\pi}\zeta\left(k_x\right)k_x \tag{33}$$

The negative sign in Eq. (33) means that the rough boundary roughness causes effective slow-down of the liquid, *i.e.,* that the coefficient $\mathcal{L}_{eff}$ (33) is the stick length rather than the slip length. In other words, there is an additional roughness-induced friction.

The condition $\Lambda\mathcal{L}_{eff}/R \ll 1$, when one can replace the rough wall by a stick-slip boundary condition is equivalent to

$$\frac{\Lambda\mathcal{L}_{eff}}{R} \sim \Lambda\epsilon^2 \sim \frac{h^2}{R\delta} \ll 1 \tag{34}$$

Surprisingly, the effective boundary condition (30),(33) , taken by itself, cannot emulate the roughness-driven attenuation (21). The reason is the presence normal flows near the boundary, $v_y\left(x\right)$, which are completely absent within the effective stick-slip description (30) , (31) in which $v_y = 0$. The attempts to modify the boundary condition (30) so that to reproduce both the velocity and attenuation correctly by, for example, introducing a two-component or complex stick-slip length, fail. In order to emulate the correct behavior of liquid near a rough wall, one has not only to introduce the stick-slip length (30),(33) , but also to renormalize the viscosity near the wall as

$$\eta_{eff}\left(y\right) = \eta\left[1+\beta\delta\left(y\right)\right], \tag{35}$$

where renormalization parameter β is equal at small Λ to

$$\beta \approx 2\left[\frac{\mathcal{L}_{eff}}{R}+\frac{\Lambda}{\sqrt{2}}\frac{h^2}{R^2}\right]. \tag{36}$$

The effective boundary conditions (30),(33) , (35),(36) are the main result of this paper. These conditions allow one to replace the random rough boundary by an equivalent problem with the flat boundary and the effective stick-slip length and renormalized viscosity. The necessity of the renormalization of the viscosity means that the rough surface slows the flow down and changes the attenuation. Usually, the slip boundary condition is understood in terms of the existence of a peculiar thin slip boundary layer with the thickness of the order of the mean free path and with the properties that are somewhat different from the rest of the liquid. In the case of the rough walls, one should not only introduce the effective stick-slip layer with the thickness that is determined by δ and R, but also to renormalize the viscosity in this layer explicitly. A simple physical model that clarifies the meaning of the effective parameters is given in Appendix B.

IV. COMPARISON FOR DIFFERENT TYPES OF RANDOM INHOMOGENEITIES

In this Section we address the question whether it is possible to extract information on the properties of the rough surface from the frequency dependence of attenuation of transverse oscillations. Statistical properties of the random surface are described by the correlation function of surface inhomogeneities, $\Xi\left(X\right) = h^2\zeta(x),\ x = X/R$, Eq. (6) . Experimentally, the correlation function can exhibit different types of long-range behavior and can assume various forms [25].

TABLE I: The position of the maximum of the funcion (37) and the value of stick-slip length Eq. (43) for different types of the surface correlation function.

♯	Correlator type	Form, $\zeta(x)$	Fourier image, $\zeta(k)$	$\Lambda_{\max}$	$-\ell_1 (\Lambda \ll 1)$
1.	Gaussian	$\exp\left(-x^2\right)$	$\sqrt{\pi}\exp\left(-k^2/4\right)$	1.293	$4/\sqrt{\pi}$
2	Power-law	$\left(1+x^2\right)^{-(\mu+1/2)}$	$\frac{\sqrt{\pi}}{2^{\mu-1}\Gamma(\mu+1/2)}\lvert k\rvert^{\mu} K_{\mu}(\lvert k\rvert)$		$\frac{4}{\sqrt{\pi}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)}$
2a	$\mu=1/2$: Lorentzian	$\left(1+x^2\right)^{-1}$	$\pi\exp(-\lvert k\rvert)$	1.320	2
2b	$\mu=3/2$: Staras	$\left(1+x^2\right)^{-2}$	$\frac{\pi}{2}(1+\lvert k\rvert)\exp(-\lvert k\rvert)$	1.825	3
3	Power-law Fourier image	$\frac{1}{2^{\nu-1}\Gamma(\nu)}\lvert x\rvert^{\nu} K_{\nu}(\lvert x\rvert)$	$2\sqrt{\pi}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu)}\left(1+k^2\right)^{-\{\nu+1/2\}}$		$\frac{2}{\sqrt{\pi}}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu)}\frac{1}{\nu-1/2}$
3a	$\nu=1/2$: exponential	$\exp(-\lvert x\rvert)$	$2\left(1+k^2\right)^{-1}$	no max	∞
3b	$\nu=3/2$	$(1+\lvert x\rvert)\exp(-\lvert x\rvert)$	$4\left(1+k^2\right)^{-2}$	1.238	$4/\pi$

Three broad classes of the correlation functions $\zeta(x)$ and their Fourier images $\zeta(k)$ (the so-called power density spectral function, or power spectra) are summarized in Table I. For better comparison, all the correlators are normalized in the same way, $\zeta(x=0)=1$. Note, that this normalization differs from the one used in Ref. [28] for conductivity of ultrathin films: the natural reference point for the conductivity was its value in the limit $kR \to 0$ and all the correlation functions in Ref. [28] have been normalized using $\zeta(k=0)=1$. For the hydrodynamic problem in this paper, the normalization $\zeta(x=0)=1$ provides a better reference.

The most commonly used correlation function, namely, the Gaussian correlator, is listed first. The next class of the correlation functions covers power-law correlators with the exponentially decaying Fourier images (power spectra), $|k|^{\mu} K_{\mu}(|k|)$. Here, the most widely used are the Lorentzian correlator (index $\mu=1/2$) and the Staras correlator ($\mu=3/2$). The third class of the correlation functions includes the conjugate correlators, namely, the exponentially decaying correlators with the power-law spectral function $\zeta(k)$. In our dimensionless notations, Eq. (5), all the correlators have the correlation radius equal to one.

The most convenient observable is the frequency dependence of the relative attenuation, Eq.(21):

$$\Delta\Gamma(\Lambda)=\frac{\Delta Q}{\Lambda\epsilon^2 Q_0}\equiv\Lambda\int_0^{\infty}\frac{dt}{\pi}\zeta(t\Lambda)\,\phi(t)\,, \tag{37}$$

$$\phi(t)=1-\sqrt{\sqrt{1+t^4}-t^2}. \tag{38}$$

In the limits $t\ll 1$ and $t\gg 1$, the function $\phi(t)$ has the following asymptotic expansions:

$$\phi(t)\approx\begin{cases} t^2/2-t^4/8, & t\ll 1\\ 1-1/\sqrt{2}t, & t\gg 1.\end{cases} \tag{39}$$

Note, that the piecewise continues function, defined by the expressions in Eq. (39) connected at the point $t=\sqrt{2}$, gives a good approximation for $\phi(t)$ in the whole range of t. This can be useful in simple approximations of the integral (37). The integral (37) can be conveniently split into two parts, $\Delta\Gamma_1$ and $\Delta\Gamma_2$, which correspond to the contributions from small and large t.

In the laminar limit, $\Lambda\ll 1$, the main contribution comes from large t :

$$\Delta\Gamma\sim\Delta\Gamma_2=\Lambda-\Lambda^2\ln(\Lambda)/\sqrt{2}+O\left(\Lambda^2\right). \tag{40}$$

The first two terms in this expression are the same for the correlators of all types. Therefore, in the low-frequency limit with large decay length it is impossible to distinguish statistical properties of different surfaces. The physical reason is obvious: large-scale attenuation processes on the scale of decay length δ are not very sensitive to the details of surface inhomogeneities with the size $R\ll\delta$.

The situation is different in the opposite case of large Λ. In this limit for Gaussian and power-law correlators with the exponential power spectra (types 1 and 2 in the Table I), the contribution from large t to the integral (37) is exponentially small. An estimate of the contribution from small k yields

$$\Delta\Gamma\sim\Delta\Gamma_1\approx\int_0^{\infty}\frac{dk}{\pi}\frac{k^2\zeta(k)}{2\Lambda}=-\frac{1}{2\Lambda}\left.\frac{d^2\zeta(x)}{dx^2}\right|_{x=0}=\frac{a}{\Lambda}, \tag{41}$$

where $a=1$ for the Gaussian correlator and $a=\mu+1/2$ for the power-law correlators.

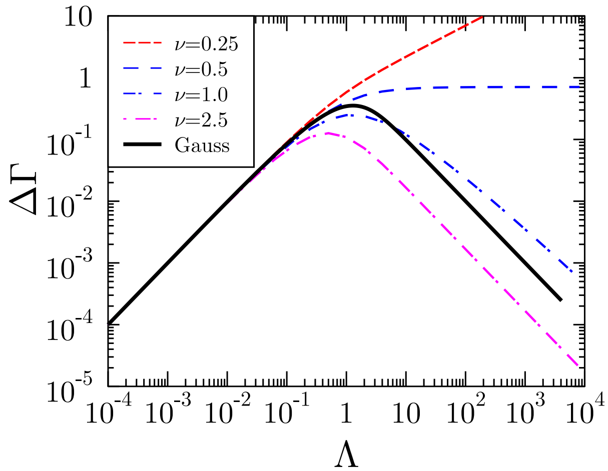

FIG. 2: Correction to the energy dissipation rate, $\Delta\Gamma$ as a function of the frequency parameter Λ for Gaussian and ν-correlators in log-log scale.

For the correlators with the power-law power spectrum (correlators of the type 3 in the Table I), the contribution from large t, $\Delta\Gamma_2$, is

$$\Delta\Gamma_2 \sim \Lambda \int_0^{1/\Lambda} \frac{t^{2\nu-1}dt}{\left(1+t^2\right)^{\nu+1/2}} \sim \Lambda^{1-2\nu}.$$

The contribution from small t, $\Delta\Gamma_1$, strongly depends on the value of the exponent ν. If $0<\nu<1$, then the value of $\Delta\Gamma_1$ is determined by the upper limit of the corresponding part of the integral and it is also proportional to $\Lambda^{1-2\nu}$. If $\nu>1$, then the first terms in the Taylor expansion for $\phi\left(t\right)$ yields a convergent integral proportional to Λ^{-1}, while the rest gives the terms with the smaller exponent $\Lambda^{1-2\nu}$:

$$\Delta\Gamma_1 \sim \int_0^{\Lambda} \frac{dk}{\left(1+k^2\right)^{\{\nu+1/2\}}}\left[\frac{k^2}{2\Lambda}-\frac{k^4}{8\Lambda^3}+...\right] \sim \frac{1}{\Lambda}+O\left(\frac{1}{\Lambda^{2\nu-1}}\right).$$

Thus, the energy dissipation rate for the correlators with the power-law power spectrum is determined by the value of the index ν :

$$\Delta\Gamma\left(\Lambda,\nu\right) \sim \begin{cases} \Lambda^{1-2\nu}, & 0<\nu<1, \\ \Lambda^{-1}, & \nu>1. \end{cases} \tag{42}$$

Comparison of the asymptotic behavior of the function $\Delta\Gamma\left(\Lambda\right)$ for small and large Λ, Eqs. (40) − (42), indicates that this function should have a maximum at $\Lambda=\sqrt{2}R/\delta\sim 1$ except for the correlators with small ν. In experiment, the position of this maximum on the frequency dependence of the attenuation can become a direct measurement of the correlation radius (size) of the surface inhomogeneities R.

The numerical results are summarized in Fig. 2 which presents the functions $\Delta\Gamma\left(\Lambda\right)$ for various correlators. Numerical values of the position of the maximum for $\Delta\Gamma\left(\Lambda\right)$ for various correlation functions are presented in the Table I.

The last column in the Table describes the dimensionless roughness-driven stick-slip length $\ell_1=\mathcal{L}_{eff}/\epsilon^2 R$, Eq. (33), for various correlators at small Λ,

$$-\ell_1\left(\Lambda\ll 1\right)=2\int_0^{\infty}\frac{dk_x}{\pi}\zeta\left(k_x\right)k_x. \tag{43}$$

V. SUMMARY

In summary, we calculated roughness-driven contributions to the hydrodynamic flows, energy dissipation, and the friction force in a wide range of parameters. We also investigated the possibility of replacing a random rough surface by a set of effective stick-slip boundary conditions on a flat surface. Such a replacement is highly desirable for analysis of experimental data and/or simplification of hydrodynamic computations in microchannels. The replacement turned out to be possible when the hydrodynamic decay length (the viscous wave penetration depth) is larger than the size of random surface inhomogeneities, Eq.(34). The effective boundary conditions contain two constants: the stick-slip length and the renormalization of viscosity near the boundary. The stick-slip length and the renormalization coefficient are expressed explicitly via the correlation function of surface inhomogeneities. The corresponding expressions are quite simple and can be easily used for analysis of experimental data or in hydrodynamic computations. The effective stick-slip length is negative meaning the effective average slow-down of the hydrodynamic flow by the rough surface (stick rather than slip motion).

For better understanding of the results, in Appendix B below we present a simple hydrodynamic model that illustrates our general hydrodynamic calculations.

In the process of the derivation of the effective boundary condition, we reduced the Navier-Stokes equation with the boundary condition on a random rough wall to the exactly equivalent closed integral equation with the homogeneous boundary condition on the ideal flat wall. All the information on the surface roughness is contained in the kernel of this integral equation. The equation can be solved by standard methods.

The effective boundary parameters are analyzed numerically for three classes of surface correlators including the Gaussian, power-law and exponentially decaying correlators. The energy dissipation near the rough surface is calculated as a function of frequency for these types of the correlation functions. The position of the maximum on the frequency dependence of the dissipation allows one to extract the correlation radius (characteristic size) of the surface inhomogeneities directly from, for example, experiments with torsional quartz oscillators. The stick-slip length is also evaluated numerically for all three classes of surface correlators.

The next step should be the evaluation of the effective stick-slip length for ultrathin flow channels of the thickness L for which L is expected to gradually replace the decay length δ in the expressions for the slip length. Another desirable development would be the incorporation into the effective slip length of both surface and bulk scattering processes beyond the simple Matthiessen's rule in the same spirit as recent calculations for helium flows in microchannels [29].

Acknowledgments

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APPENDIX A: SOLUTION OF THE NAVIER-STOKES EQUATION FOR FLUIDS RESTRICTED BY RANDOM ROUGH WALLS

First, we assume that all the variables have the harmonic time dependence, $\exp(-i\omega t)$, transform the linearized Navier-Stokes equation (7) to the non-inertial coordinate frame in which the wall is at rest,

$$v_x \rightarrow v_x - \exp(-i\omega t), \; x \rightarrow x - \int \exp(-i\omega t)\, dt,$$

and introduce the stream function $\psi(x, y)$ as

$$v_x = \frac{\partial \psi}{\partial y}, \; v_y = -\frac{\partial \psi}{\partial x}. \tag{A1}$$

In this reference frame, the Navier-Stokes equation (7) and the boundary condition (3) can be rewritten as the following equation for the stream function:

$$-i\Lambda^2 \nabla^2 \psi - \nabla^4 \psi = 0, \tag{A2}$$

$$\frac{\partial \psi(x, \epsilon\xi(x))}{\partial y} = 1, \; \frac{\partial \psi(x, \epsilon\xi(x))}{\partial x} = 0, \tag{A3}$$

$$\psi(x, \infty) = const. \tag{A4}$$

The difficulty in solving Eqs. $(A2)-(A4)$ originates from the presence of a random function $\xi(x)$ in the boundary condition. The next step is the coordinate transformation

$$x \rightarrow x,\ y \rightarrow y - \epsilon\xi(x), \tag{A5}$$

that flattens the wall, making the boundary condition (12) simple,

$$\frac{\partial\psi(x,0)}{\partial y} = 1,\ \frac{\partial\psi(x,0)}{\partial x} = \epsilon\xi_x(x),\ \psi(x,\infty) = const. \tag{A6}$$

The change in derivatives introduces the additional term $\widehat{V}(\xi,\partial_x)\psi$ into the $r.h.s$ of Eq.$(A2)$,

$$-i\Lambda^2\nabla^2\psi - \nabla^4\psi = \widehat{V}(\xi,\partial_x)\psi \tag{A7}$$

where

$$\hat{V} = \epsilon\widehat{V}_1 + \epsilon^2\widehat{V}_2 + \epsilon^3\widehat{V}_3 + \epsilon^4\widehat{V}_4 \tag{A8}$$

$$\hat{V}_1\psi = -\left(2\lambda^2\xi_x\psi_{yx} + \lambda^2\xi_{xx}\psi_y + 4\xi_{xxx}\psi_{yx} + 6\,\xi_{xx}\psi_{yxx} + \xi_{xxxx}\psi_y + 2\xi_{xx}\psi_{yyy} + 4\xi_x\psi_{yxxx} + 4\xi_x\psi_{yyyx}\right),$$

$$\hat{V}_2\psi = \left(\lambda^2{\xi_x}^2\psi_{yy} + 4\xi_{xxx}\xi_x\psi_{yy} + 12\xi_x\xi_{xx}\psi_{yyx} + 6{\xi_x}^2\psi_{yyxx} + 2{\xi_x}^2\psi_{yyyy} + 3{\xi_{xx}}^2\psi_{yy}\right),$$

$$\hat{V}_3\psi = -2{\xi_x}^2\left(3\xi_{xx}\psi_{yyy} + 2\xi_x\psi_{yyyx}\right),$$

$$\hat{V}_4\psi = {\xi_x}^4\psi_{yyyy},$$

and lower indices denote the differentiation of the functions ξ and ψ.

The simplicity of boundary conditions in new coordinates allows us to find the Green's function $G(x-x',y,y')$ for the operator in the *l.h.s.* of Eqs. $(A2)$, $(A7)$. With the help of this Green's function, our initial problem with a boundary condition on the random rough surface reduces to the compact integral equation:

$$\psi(k_x,y) = \psi_{inh}(k_x,y) + \int_0^\infty dy'\,G(k_x,y,y')\int_{-\infty}^{\infty}\frac{dk_x'}{2\pi}\widehat{V}(k_x-k_x',y')\,\psi(k_x',y'), \tag{A9}$$

where we performed the Fourier transformation in x-direction (in the new coordinate frame, the geometry of the boundary is independent of x), and

$$\psi_{inh}(k_x,y) = \int_{-\infty}^{\infty} dx e^{-ik_x x}\psi(x,y) \tag{A10}$$

is a solution of Eq. $(A2)$ with $\widehat{V}=0$ and boundary conditions $(A3),(A4)$. With this definition of $\psi_{inh}(k_x,y)$, the Green's function satisfies the homogeneous boundary conditions on the wall. Note, that Eq. $(A9)$ is an *exact* equivalent of our initial problem with the random rough wall and, in principle, can be solved for an arbitrary function $\xi(x)$.

The function $\psi_{inh}(k_x,y)$ is determined by the characteristic equation for the operator in *l.h.s.* of Eq. $(A2)$

$$s^4 - \left(2k_x^2 - i\Lambda^2\right)s^2 - \left(ik_x^2\Lambda^2 - k_x^4\right) = 0. \tag{A11}$$

This equation has four solutions:

$$\begin{aligned} k_y &= \pm s_1, \pm s_2; \\ s_1 &= -|k_x|,\ s_2 = \sqrt{k_x^2 - i\Lambda^2} \equiv -\alpha + i\beta, \\ \alpha,\beta &= \frac{1}{\sqrt{2}}\sqrt{\sqrt{k_x^4+\Lambda^4} \pm k_x^2} \geq 0. \end{aligned} \tag{A12}$$

We are interested only in the functions $\psi_{inh}(k_x,y)$ that decrease at $y\rightarrow\infty$. Therefore, the general solution of the homogeneous Eq.$(A2)$ with the boundary condition $(A3)$ has the form

$$\psi_{inh}(k_x,y) = \frac{2\pi}{i\lambda}\delta(k_x)\left[e^{i\lambda y} - 1\right] + \frac{\epsilon\xi(k_x)}{s_2 - s_1}\left[s_2 e^{s_1 y} - s_1 e^{s_2 y}\right] \tag{A13}$$

and contains the contribution without ϵ, $\psi_0(k_x, y)$, and the term linear in ϵ. Similar calculations yield the Green's function:

$$G(k_x, y, y') = \frac{1}{2i\Lambda^2}\left[\frac{1}{s_2}\left(e^{s_2|y-y'|} - e^{s_2(y+y')}\right) - \frac{1}{s_1}\left(e^{s_1|y-y'|} - e^{s_1(y+y')}\right)\right] + \frac{1}{i\Lambda^2(s_2 - s_1)}\left[e^{s_1(y+y')} + e^{s_2(y+y')} - e^{s_1 y + s_2 y'} - e^{s_1 y' + s_2 y}\right]. \tag{A14}$$

The last result can be also obtained by noticing that our Green's function is proportional to the difference between the Green's functions for the two-dimensional Laplace and Helmholtz equations with the same boundary conditions:

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') &= \lambda^{-2}(G_L - G_H), \\ G_L(\mathbf{r}, \mathbf{r}') &= -\frac{1}{2\pi}\ln(R_S/R_I), \ G_2(\mathbf{r}, \mathbf{r}') = \frac{i}{4}\left[H_0^{(1)}(\lambda R_S) - H_0^{(1)}(\lambda R_I)\right], \\ R_{S,I} &= \sqrt{(x-x')^2 + (y \mp y')^2}. \end{aligned}$$

In our case of slight roughness, it is sufficient to find only the first three terms of the expansion of the stream function ψ, Eq. $(A9)$, in powers of the small parameter ϵ, $\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots$ Since all the terms in the operator $\widehat{V}$ contain ϵ, the only part of ψ without ϵ is the first term in Eq. $(A13)$ for ψ_{inh},

$$\psi_0(k_x, y) = \frac{2\pi}{i\lambda}\delta(k_x)\left[\exp(i\lambda y) - 1\right]. \tag{A15}$$

The first order term in ψ contains the remaining part of ψ_{inh} and the first order term in the integral $(A9)$ with

$$\hat{V}_1(k_x, \partial_y)\psi_0 = -\xi(k_x)\left[k_x^2\lambda^2 + k_x^4\right]e^{i\lambda y}$$

Integration gives

$$\psi_1(k_x, y) = \xi(k_x)\left[e^{i\lambda y} + \frac{i\lambda}{s_2 - s_1}\left(e^{s_1 y} - e^{s_2 y}\right)\right]. \tag{A16}$$

The calculation of the second order term requires straightforward integration for much more cumbersome expressions. However, the general expression for ψ_2 is not required for further calculations; it is sufficient to have only the expression for ψ_2 averaged over the surface inhomogeneities, $\langle\psi_2\rangle$. The resulting expression for the stream function contains products of the derivatives of the surface profile $\xi^{(n)}(x)$. These products should be averaged over surface inhomogeneities using the definition of the correlation function $\zeta(x)$, Eqs. (6)

$$\left\langle \xi^{(n)}(x)\,\xi^{(m)}(x')\right\rangle = (-1)^m\,\zeta^{(n+m)}(x - x'). \tag{A17}$$

In the end, after substantial cancellations that accompany the averaging,

$$\langle\psi_2(k_x, y)\rangle = \delta(k_x)\int_{-\infty}^{\infty} dk'_x\,\zeta(k'_x)\left[\frac{s_1 + s_2}{i\lambda}\left(i\lambda e^{i\lambda y} + s_1 e^{s_1 y} - s_2 e^{s_2 y}\right)\right]. \tag{A18}$$

Reversing the coordinate transformation of Eq. $(A5)$ and performing the related re-expansion in ϵ,

$$\langle v_x(k_x, y - \epsilon\xi)\rangle \simeq v_x^{(0)}(y)\,\delta(k_x) + \epsilon^2\left[\left\langle v_x^{(2)}(k_x, y)\right\rangle - \left\langle \xi\frac{\partial}{\partial y}v_x^{(1)}(k_x, y)\right\rangle + \left\langle\frac{\xi^2}{2}\right\rangle\frac{\partial^2}{\partial y^2}v_x^{(0)}(y)\,\delta(k_x)\right],$$

we get for the average velocity

$$\langle v_x(x, y)\rangle = \exp(i\lambda y)\left[1 + i\lambda\varepsilon^2\int_0^{\infty}\frac{dk_x}{\pi}\zeta(k_x)\left\{s_1 + s_2 - i\lambda/2\right\}\right], \tag{A19}$$

where s_1, s_2 are given by Eq. $(A12)$.

The above equations for the stream function allow one to calculate the roughness-driven correction to the dissipation of energy and effective friction.

Time average of the bulk dissipation per unit area of the wall can be expressed via the stream function ψ as

$$Q=-\frac{\eta u_0^2}{2R}\frac{1}{A}\int dV\left\langle\overline{\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2}\right\rangle=-\frac{\eta u_0^2}{R}\frac{1}{A}\int dV\left\langle 4\overline{\Psi_{xy}^2}+\overline{\left(\Psi_{yy}-\Psi_{xx}\right)^2}\right\rangle, \tag{A20}$$

where $\Psi\left(\mathbf{r},t\right)=\mathrm{Re}\left[\psi\left(\mathbf{r}\right)e^{-i\omega t}\right]$, the overline denotes the time average over the period of oscillations, and $\langle ...\rangle$ stands for the statistical average over the random surface inhomogeneities. The time average $\overline{\Psi_{ik}^2}=\frac{1}{2}\psi_{ik}\psi_{ik}^*$.

After the coordinate transformation (A5), the attenuation up to the second order term in ϵ reduces to

$$Q=-\frac{\eta u_0^2}{2R}\int\limits_0^\infty dy\left[Q^{(0)}+\epsilon^2 Q^{(2)}\right], \tag{A21}$$

$$Q^{(0)}=\left|\psi_{yy}^{(0)}\right|^2=\Lambda^2 e^{-\sqrt{2}\Lambda y},$$

$$Q^{(2)}=\left\langle\left|\psi_{yy}^{(1)}+\xi_{xx}\psi_y^{(0)}-\psi_{xx}^{(1)}\right|^2+2\left|\xi_x\psi_{yy}^{(0)}-\psi_{xy}^{(1)}\right|^2+2\left|\psi_{xy}^{(1)}\right|^2+2\,\mathrm{Re}\left[\psi_{yy}^{(0)*}\left(\psi_{yy}^{(2)}+\xi_{xx}\psi_y^{(1)}\right)\right]\right\rangle.$$

Finally, we get

$$Q=-\frac{\eta u_0^2}{2R}\frac{\Lambda}{\sqrt{2}}\left\{1+\epsilon^2\Lambda\int\frac{dk_x}{2\pi}\zeta\left(k_x\right)\left[\Lambda-\sqrt{\sqrt{k_x^4+\Lambda^4}-k_x^2}\right]\right\}. \tag{A22}$$

The friction force acting on the area unit of the surface is

$$\mathbf{F}=\frac{\eta u_0}{R}\mathbf{f},\ f_i=-\pi_{ik}n_k, \tag{A23}$$

$$\pi_{ik}=\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)_{y=\epsilon\xi},\ \mathbf{n}=\frac{1}{\sqrt{1+\epsilon^2\xi_x^2}}\begin{pmatrix}\epsilon\xi_x\\ -1\end{pmatrix} \tag{A24}$$

Here $\mathbf{n}$ is the unit vector normal to the surface and directed out of the liquid. The square of this force is

$$f^2=f_x^2+f_y^2=\pi_{xy}^2+\frac{\pi_{yy}^2+\epsilon^2\xi'^2\pi_{xx}^2}{1+\epsilon^2\xi'^2}, \tag{A25}$$

or, via the stream function,

$$f^2=\left[\left(\psi_{yy}-\psi_{xx}\right)^2+4\psi_{xy}^2\right]\Big|_{y=\varepsilon\xi}. \tag{A26}$$

In new coordinates $(A5)$, this expression reduces to

$$\left\langle f^2\right\rangle=\left\langle\left(1+2\epsilon^2\xi_x^2\right)\psi_{yy}^2\left(x,0\right)\right\rangle.$$

After separating the real and imaginary parts and expanding in ϵ, we finally get

$$\left\langle f^2\right\rangle=\frac{\Lambda^2}{2}\left(1+\epsilon^2\int\limits_0^\infty\frac{dk_x}{\pi}\zeta(k_x)\left[\Lambda-\sqrt{\sqrt{k_x^4+\Lambda^4}-k_x^2}\right]^2\right) \tag{A27}$$

Note, that in this problem the friction force introduced by equation $(A23)$ does not determine, after integration over the surface, the full energy dissipation. In the case of inhomogeneous rough boundaries there is an additional dissipative contribution related to the term with pressure, Pn_i in the expression for the full force acting on the unit area of the surface. If one defines the friction force not via the stress tensor, Eq. $(A23)$, but assumes the experimental definition according to $F=-Q/u_0$, then the roughness-driven correction to the friction force will be given by Eq. $(A22)$ rather than by Eq. $(A27)$. Another anomaly of this problem is that one should always take into account both, x and y components of the friction force.

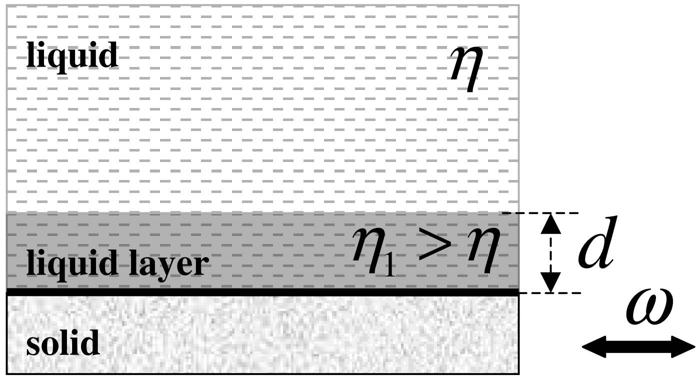

FIG. 3: Schematic geometry of the problem.

APPENDIX B: TWO-LAYER MODEL

The necessity of using two parameters in the effective boundary condition instead of a single stick-slip length can be illustrated by the following simple model. Let us consider tangential oscillations of viscous liquid which is separated form a solid substrate by a layer of another liquid with a slightly higher viscosity $\eta_1 > \eta$ and the same density (see Fig. 3). In effect, we model a rough surface by a layer of viscous liquid with somewhat different properties than in the bulk. The model has two parameters: the thickness of the layer d, and dimensionless ratio γ,

$$\gamma = \frac{\eta\lambda}{\eta_1\lambda_1} \equiv \frac{\Lambda_1}{\Lambda} \lesssim 1.$$

Assuming that the velocity in both liquids is proportional to $\exp\left(-i\omega t\right)$, we get the following equations of motion:

$$\begin{gathered}
-i\omega v_1 - \nu_1 \frac{d^2 v_1}{d\,y^2} = 0,\;\; -i\omega v - \nu \frac{d^2 v}{d\,y^2} = 0, \\
v_1\left(0\right) = 1,\;\; v_1\left(d\right) = v\left(d\right), \\
\eta_1 \frac{dv_1\left(d\right)}{d\,y} = \eta \frac{dv\left(d\right)}{d\,y}.
\end{gathered}$$

The solution is

$$\begin{aligned}
v_1\left(y\right) &= Ae^{i\lambda_1\left(y-d\right)} + Be^{-i\lambda_1\left(y-d\right)}, \\
v_2\left(y\right) &= Ce^{i\lambda y}.
\end{aligned}$$

with

$$\begin{aligned}
A, B &= Ce^{i\lambda d}\frac{1 \pm \gamma}{2}, \\
C &= \frac{e^{-i\lambda d}}{\cos\left(\lambda_1 d\right) - i\gamma \sin\left(\lambda_1 d\right)}.
\end{aligned}$$

Time average of the rate of the energy dissipation per unit are consists of contributions from both liquids:

$$\begin{aligned}
Q &= Q_I + Q_{II}, \\
Q_I &= -\eta_1 \int_0^d dy \overline{\left[\mathrm{Re}\left(\frac{\partial v_1}{\partial y} e^{-i\omega t}\right)\right]^2} \\
&= -\left|Ce^{i\lambda d}\right|^2 \eta_1 \Lambda_1 \frac{1}{8\sqrt{2}} \left[\left(1+\gamma\right)^2 \left(e^{\sqrt{2}\lambda_1 d} - 1\right) + \left(1-\gamma\right)^2 \left(1 - e^{-\sqrt{2}\lambda_1 d}\right) + 2\left(\gamma^2 - 1\right)\sin\left(\sqrt{2}\lambda_1 d\right)\right], \\
Q_{II} &= -\eta \int_d^\infty dy \overline{\left[\mathrm{Re}\left(\frac{\partial v_2}{\partial y} e^{-i\omega t}\right)\right]^2} = -\left|Ce^{i\lambda d}\right|^2 \frac{\eta\Lambda}{2\sqrt{2}}.
\end{aligned}$$

If the thickness of the layer d is smaller than the decay length δ, $\lambda_1 d \ll 1$,

$$\begin{aligned} C &\to 1 - e^{i3\pi/4}\Lambda d\left(1-\gamma^2\right), \\ \left|Ce^{i\lambda d}\right|^2 &\to 1 - \sqrt{2}\Lambda d\gamma^2 + \Lambda^2 d^2\gamma^4, \\ v\left(y\right) &\approx e^{i\lambda y}\left[1 - i\lambda d\left(1-\gamma^2\right)\right] \\ -Q &\approx \frac{\eta\Lambda}{2\sqrt{2}}\left[1 + \Lambda^2 d^2\gamma^2\left(1-\gamma^2\right)\right]. \end{aligned}$$

Note, that the condition $\lambda_1 d \ll 1$ does not necessarily mean that the layer by itself is thin.

The last two equations show that in this limit

$$v\left(y \geq d\right) \approx \mathrm{Re}\left\{u_0 e^{i(\lambda y - \omega t)}\left[1 - e^{i3\pi/4}\Lambda d\left(1-\gamma^2\right)\right]\right\}, \tag{B1}$$

$$Q \approx \frac{\eta u_0^2\Lambda}{2\sqrt{2}}\left[1 + \Lambda^2 d^2\gamma^2\left(1-\gamma^2\right)\right]. \tag{B2}$$

Comparison of Eqs.($B1$),($B2$) with Eqs. (33)-(36) gives the mapping of the effective viscous layer model onto the problem with a rough surface:

$$\begin{aligned} -d\left(1-\gamma^2\right) &= \mathcal{L}_{eff}/R \equiv \epsilon^2\ell_1, \\ d^2\gamma^2\left(1-\gamma^2\right) &= \epsilon^2\ell_2, \end{aligned}$$

where ℓ_1, ℓ_2 are given by the low-frequency equations (24),(25).

In this limit $d \ll \delta$, the contribution of the layer to the dissipation, Q_I, corresponds to the $\delta-$type renormalization of the viscosity in the effective boundary condition of Section III with renormalization parameter

$$\beta = \epsilon^2\left[2\ell_1 + \sqrt{2}\Lambda\ell_2\right].$$

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